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ON REALIZATION OF TANGENT CONES OF HOMOLOGICALLY AREA-MINIMIZING COMPACT SINGULAR SUBMANIFOLDS

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Abstract. We show that every area-minimizing hypercone and every oriented Lawlor cone in [Law91] can be realized as a tangent cone at a point of some homologically area-minimizing singular compact submanifold. In particular this generalizes the result of N. Smale [Sma99].

1. Introduction

Let C be a k-dimensional cone over link $L \subset S^{n-1}(1)$ in an Euclidean space $(\mathbb{R}^n, g_E)$. We call C area-minimizing (mass-minimizing) if $C_1 = C \bigcap \mathbf{B}^n(1)$ has least mass among all integral (normal) currents (see [FF60]) with boundary L. We say that a d-closed compactly supported integral current in a Riemannian manifold is homologically area-minimizing (mass-minimizing) if it has least mass in its homology class of integral (normal) currents.

A well-known result of Federer (Theorem 5.4.3 in [Fed69], also see Theorem 35.1 and Remark 34.6 (2) in Simon [Sim83]) asserts that a tangent cone at a point of an area-minimizing rectifiable current is itself area-minimizing. This paper studies its converse realization question by compact submanifolds ($\star$):

> *Can any area-minimizing cone be realized as a tangent cone at a point of some homologically area-minimizing* compact *singular submanifold?*

Through techniques of geometric analysis and Allard's regularity theorem, N. Smale found realizations for all strictly minimizing, strictly stable hypercones (see [HS85]) in [Sma99]. They are first examples of codimension one homological area-minimizers with singularities.

Very recently, different realizations of many area-minimizing cones, including all homogeneous minimizing hypercones (classified by Lawlor [Law91], also see [Law72] and [Zhab]) and special Lagrangian cones, by extending local calibration pairs were discovered in [Zha].

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However in general the answer to ($\star$) is still far to be known. In this paper, we focus on two important classes of mass-minimizing cones – minimizing hypercones [1] and oriented Lawlor cones.

For hypercones, two long-term standing conjectures (or equivalent versions) raised by Simon, Hardt and Simon respectively are the followings.

Conjecture 1.1. *Except trivial examples in low dimensions, all minimizing hypercones are strictly area-minimizing?*

Conjecture 1.2. *Any non-trivial strictly area-minimizing hypecone is always strictly stable?*

Up to now it is unclear how far it is for a minimizing hypercone to be strictly stable and strictly area-minimizing. An important characterization of minimizing hyercones in [HS85] is that each of them possesses a canonical singular "calibration".

By Lawlor cones we mean area-minimizing cones shown in [Law91]. He studied when certain preferred bundle structure (somehow analogous to that in [HS85] for hypercones, nevertheless involving curvatures more heavily without the limitation to codimension one) of some angular neighborhood of a minimal cone exists, and successfully added quite a few interesting new oriented area-minimizing cones (and non-orientable area-minimizing cones in the sense of modulo 2 as well). In the oriented case, such bundle structure naturally induces a "calibration" of the cone that is singular in a set of codimension one and possibly also along the cone.

By virtue of these peculiar calibrations of minimizing hypercones and oriented Lawlor cones, we obtain realizations for them.

Theorem 1.3. *Every minimizing hypercone can be realized to* ($\star$).

Remark 1.4. *Our construction removes the requirements of a minimizing hypercone's being strictly stable and being strictly minimizing in* [Sma99]. *Hence the case of codimension one is completely settled.*

Theorem 1.5. *Every oriented Lawlor cone can be realized to* ($\star$).

Remark 1.6. *This answers affirmatively to* ($\star$) *for lots of area-minimizing cones of higher codimensions, for instance, a minimal cone C over a product of two or more spheres satisfying (1)* $\dim(C) > 7$, *or (2)* $\dim(C) = 7$ *with none of the spheres being a circle (cf. Theorem 5.1.1 in* [Law91]*). These cones do not split. Namely, they cannot be written as products of two or more area-minimizing cones of lower dimensions (vs. N. Smale* [Sma00]*).*

[1] By [Fed74] or [Mor86], the area-minimality of a hypercone is equivalent to its mass-minimality. So we say minimizing for short.

The paper is organized as follows. In §2 our preferred model S of construction is introduced. By a monotonicity result of Allard, we get Lemma 3.1 which helps us transform the global realization question to a local problem around S in §4. Thus, we only need to construct a smooth metric $\bar{g}$ on some neighborhood $\tilde{U}$ of S such that S is homologically area-minimizing in $\tilde{U}$.

We discuss the case of codimension one in §5. There are two steps. First suitably extend the canonical (local, singular and non-coflat) calibrations around p_1 and p_2 (see §2) to a C^1 closed form Φ in a neighborhood $\tilde{U}$ of S. Then a smooth metric $\tilde{g}$ can be created to make Φ a C^1 calibration of S. Hence we gain the homological area-minimality of S in $\tilde{U}$.

In §6 realizations of oriented Lawlor cones are constructed. The idea is roughly the same. However the calibration is discontinuous in a set of codimension one. So we consider its regularization through convolution for the desired local homological area-minimality of S. Although the approximating closed forms may have comass greater than one somewhere, by the mildness of calibrations in [Law91] and Lebesgue's bounded convergence theorem, the needed area-minimality can be achieved.

2. Model of Construction

Given a k-dimensional cone $C \subset \mathbb{R}^N$. As in [Sma99], consider $\Sigma_C \triangleq (C \times \mathbb{R}) \bigcap S^N(1)$ in $\mathbb{R}^{N+1}$. Let M be an embedded oriented connected compact k-dimensional submanifold in some N-dimensional oriented compact manifold T with $[M] \neq [0] \in H_k(T;\mathbb{Z})$. Within smooth balls round a point of M and a regular point of Σ_C respectively one can connect T and $S^N(1)$, M and Σ_C simultaneously through one connected sum. Denote by X and S the resulting manifold and submanifold (singular at two points p_1 and p_2). Apparently $[S] \neq [0] \in H_k(X;\mathbb{Z})$.

3. Positive Lower Bound of Mass

The lemma below will play a key role in §4.

Lemma 3.1. *Let g be a metric on a compact manifold X, $W \Subset X$ an open domain where $\overline{W}$ forms a manifold with nonempty boundary $\partial\overline{W}$, and α a positive number. Then there exists $\beta = \beta_{\alpha, g|_{\overline{W}}} > 0$ such that every rectifiable current K in W with no boundary, vanishing generalized mean curvature vector field δK and at least one point in its support α away from $\partial\overline{W}$ has mass greater than β.*

Proof. By Nash's embedding theorem [Nas56], $(\overline{W}, g|_{\overline{W}})$ can be isometrically embedded through a map f into some Euclidean space $(\mathbb{R}^s, g_E)$. Then $f_\# K$ is a rectifiable current of $f(\overline{W})$. Denote the induced varifold by $V_{f_\# K}$. Since K has no boundary in W and δK vanishes, the norm of $\delta V_{f_\# K}$ in $\mathbb{R}^s$ is bounded from above a.e. by a constant A depending upon f only.

Let $\overline{W_\alpha} = \{x \in W : \mathrm{dist}_g(x, \partial\overline{W}) \geq \alpha\}$. Define $2\mu = \mathrm{dist}_{g_E}(f(\overline{W_\alpha}), f(\partial\overline{W}))$. Note that the density of $V_{f_\# K}$ is a.e. at least one on the support $\mathbf{spt}(f_\# K)$ of $f_\# K$. Therefore there exists some point $p \in \mathbf{spt}(f_\# K) \bigcap f(W)$ with $\lambda \triangleq \mathrm{dist}_{g_E}(p, f(\partial\overline{W})) > \mu$ and density at least one.

By applying the following monotonicity result of Allard to A, p, μ and U the open λ-ball centered at p, we obtain our statement.

Theorem 3.2 ([All72]). *Suppose* $0 \leq A < \infty$, $p \in$ *support of* $\|V\|$, $V \in \mathbf{V}_m(U)$, *where* U *is an open region of* $\mathbb{R}^s$. *If* $0 < \mu < dist_{g_E}(p, \mathbb{R}^s - U)$ *and*

$$\|\delta V\|\mathbf{B}(p,r) \leq A\|V\|\mathbf{B}(p,r) \quad \textit{whenever } 0 < r \leq \mu,$$

then $r^{-m}\|V\|\mathbf{B}(p,r)\exp Ar$ *is nondecreasing in* r *for* $0 < r \leq \mu$.

□

4. Reduction of (★) from Global to Local

The following theorem indicates that the essential difficulty of (★) comes from local. Hence in §5 and §6 we make constructions on some neighborhood of S only.

Theorem 4.1. *Suppose* S *is homologically area-minimizing in* $(U, \bar{g})$ *where* U *is an open neighborhood of* S *and* $\bar{g}$ *is a smooth metric on* U. *Then there exists a smooth metric* $\hat{g}$ *on the compact manifold* X *such that* S *is homologically area-minimizing in* $(X, \hat{g})$.

Proof. Take open neighborhoods W, W' and W'' of S so that $W'' \Subset W' \Subset W \Subset U$ and the closer of W (W' and W'' respectively) is a manifold with nonempty boundary. Extend $\bar{g}$ to a metric $\tilde{g}$ on X with

$$\tilde{g}|_W = \bar{g}|_W.$$

Set $\alpha = \mathrm{dist}_{\tilde{g}}(\partial\overline{W'}, \partial\overline{W})$. Let β be the lower bound in Lemma 3.1 for α, domain $\overline{W'}^c$ and $\tilde{g}|_{\overline{W'}^c}$. Choose $\gamma = (t\beta^{-1}\mathrm{Vol}_{\tilde{g}}(S))^{-\frac{2}{k}} < 1$ for some large constant $t > 1$. Then construct $\hat{g}$ as follows.

$$\hat{g} = \begin{cases} \gamma\tilde{g} & \text{on } W'' \\ h\tilde{g} & \text{on } W'' \sim W' \\ \tilde{g} & \text{on } X \sim W' \end{cases} \tag{4.1}$$

where h is a smooth function on $\overline{W'} \sim W''$, no less than γ and equal to one near $\partial\overline{W'}$.

Now we show that S is homologically area-minimizing in $(X, \hat{g})$.

By the celebrated compactness result in Federer and Fleming [FF60] there exists an area-minimizing current T in $[S]$ with nonempty $\mathbf{spt}T$.

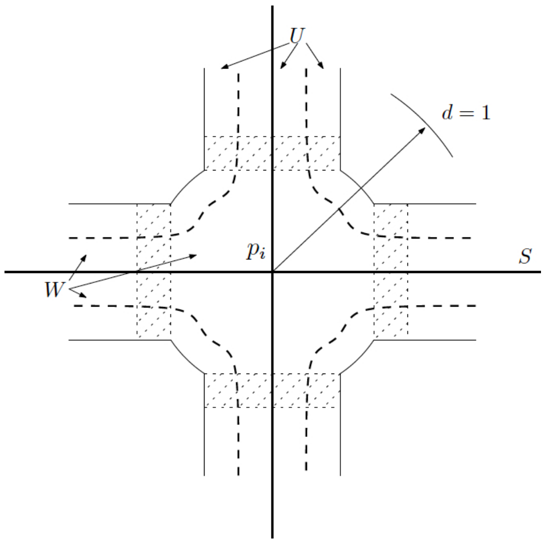
Case One: $\mathbf{spt}T$ is not contained in W. According to our construction, $\mathbf{M}(S) = \frac{\beta}{t} < \beta < \mathbf{M}(T)$ by Lemma 3.1. Contradiction.

Case Two: $\mathbf{spt}T \subset W$. By assumption and (4.1) S is homologically area-minimizing in $(W, \hat{g}|_W)$. As a result, S and T share the same mass. Hence S is homologically area-minimizing in $(X, \hat{g})$. □

Remark 4.2. $[S] \neq [0] \in H_k(X; \mathbb{Z})$ *is crucial in our proof.*

5. Realization of Minimizing Hypercones

Choose a metric g for our model in §2 such that
(i). balls $\mathbf{B}^g_{p_i}(1)$ of radius one centered at p_i are disjoint, and
(ii). local model $S \bigcap \mathbf{B}^g_{p_i}(1)$ in $(\mathbf{B}^g_{p_i}(1), g|_{\mathbf{B}^g_{p_i}(1)})$ is exactly C_1 in $(\mathbf{B}^N(1), g_E|_{\mathbf{B}^N(1)})$.



Now take U to be an open neighborhood of S shown in the picture.

Let us recall a beautiful result due to Hardt and Simon.

Theorem 5.1 (Theorem 2.1 in [HS85])**.** *Assume C is an area-minimizing hypercone in $\mathbb{R}^N$. If E is either one of the components E_+, E_- of $\mathbb{R}^N \sim C$, then there is a unique oriented connected embedded real analytic minimizing hypersurface $H \subset E$ with $H = \partial[[F]]$, $\overline{F} \subset \overline{E}$, F open, the singular set of H empty and the distance of H and the origin equal to one. Moreover, H has the property that for any $\xi \in E$ the ray $\{t\xi : t > 0\}$ intersects H in a single point.*

Hence $E_\pm$ is foliated by $\Gamma_\pm = \{tH_\pm : t > 0\}$. Let $X_\pm$ be the oriented unit normal vector of $\Gamma_\pm$ with limit ν_C (pointing into E_+) along $C \sim 0$, and $\phi_\pm$ the oriented volume form of $\Gamma_\pm$. On $\mathbb{R}^{N+1} \sim 0$, define

$$\phi = \begin{cases} \phi_+ & \text{in } E_+ \\ \lim \phi_+ (= \lim \phi_-) & \text{in } C \sim 0 \\ \phi_- & \text{in } E_- \end{cases}$$

According to [HS85], outside some large ball, each $H_\pm$ is a graph of some C^2 function on C, so ϕ is C^1 along $C \sim 0$ and smooth elsewhere.

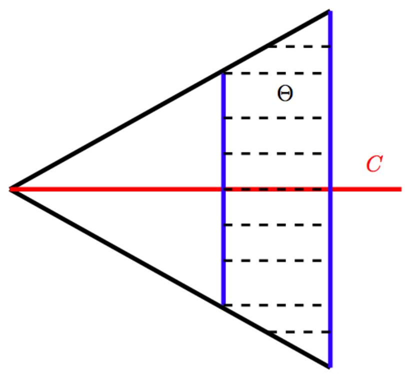
Our strategy is the following.

Step 1: glue such forms around p_1 and p_2 to a form Φ in some neighborhood of S.

Step 2: construct a smooth metric on the neighborhood so that Φ is a singular calibration of S.

In this way a realization of a minimizing hypercone can be produced based upon §4.

Assume, for some $0 < 3R < 1$, $\mathbf{B}_{p_i}(3R) \subset U$. Let $\mathbf{r}$ be the distance to the origin along C and Θ a small angular neighborhood over $C \bigcap \{1.4R < \mathbf{r} < 2R\}$ shown in the figure.



Set ω to be the unit volume form of the link L of C and $\psi = \mathbf{r}\omega$. Then $d\psi$ is the oriented unit N-dimensional form of $C \sim 0$. Since $div\ X_\pm = 0$, one has (shrink Θ if necessary)

$$\phi|_\Theta = [\pi^* d\psi]|_\Theta = [d(\pi^*\psi)]|_\Theta,$$

where π is the projection along $X_\pm$. On Θ, let ϖ be the projection to the nearest point on C and $\mathbf{r} = \mathbf{r}(\varpi(\cdot))$. Define

$$\Phi = d[\tau(\mathbf{r})(\pi^*\psi) + (1 - \tau(\mathbf{r}))(\varpi^*\psi)],$$

where τ is a decreasing smooth function from value one to zero on $[1.4R, 2R]$ with the support of $d\tau$ contained in $[1.6R, 1.7R]$. Note that Φ is the unit volume form of the cone in $\{1.4R < \mathbf{r} < 2R\} \bigcap C$.

For **Step 2**, we do some estimate on Φ. Let V be the parallel extension of v_C along fibers of ϖ, $V^\perp$ the oriented unit N-vector perpendicular to V. Then on $\overline{E_+ \bigcap \Theta}$

$$\begin{aligned}
&L_V\Phi\\
=&L_V d[\tau(\mathbf{r})(\pi^*\psi) + (1-\tau(\mathbf{r}))(\varpi^*\psi)]\\
=&d[L_V(\tau(\mathbf{r})(\pi^*\psi) + (1-\tau(\mathbf{r}))(\varpi^*\psi))]\\
=&d[\tau(\mathbf{r})L_V(\pi^*\psi)] + d[(1-\tau(\mathbf{r}))L_V(\varpi^*\psi))]\\
=&(d\tau(\mathbf{r}))\wedge[i_V(d(\pi^*\psi)) + d(i_V(\pi^*\psi))] + \tau(\mathbf{r})d[L_V(\pi^*\psi)]\\
=&(d\tau(\mathbf{r}))\wedge[i_V\phi + \pi^*d(i_{\pi_*V}\psi)] + \tau(\mathbf{r})[L_V\phi]
\end{aligned} \tag{5.1}$$

Note that

$$\varpi_*(V^\perp) = [1 + O(\mathbf{d}^2_{g_E})]V^\perp|_C \tag{5.2}$$

for the minimal cone C, where $\mathbf{d}_{g_E}(\cdot)$ is the Euclidean distance to C. Consequently,

$$(L_V V^\perp)|_C = 0. \tag{5.3}$$

Therefore by (5.1) and (5.3)

$$(L_V[\Phi(V^\perp)])|_C = (L_V\Phi)|_C(V^\perp|_C).$$

By the foliation structure, it follows from (5.1) that

$$(L_V\Phi)|_C = \tau(\mathbf{r})[L_V\phi]|_C.$$

Since ϕ is a calibration, we obtain

$$(L_V[(\Phi(V^\perp)])|_C = \tau(\mathbf{r})(L_V[(\phi(V^\perp)])|_C \leq 0. \tag{5.4}$$

The same argument on $\overline{E_- \bigcap \Theta}$ produces

$$(L_{-V}[(\Phi(V^\perp)])|_C = \tau(\mathbf{r})(L_{-V}[(\phi(V^\perp)])|_C \leq 0. \tag{5.5}$$

Hence, (5.4), (5.5) and the compactness of $[1.4R, 2R]$ imply that there exists a positive constant K such that in a sufficiently small neighborhood Ξ of $C\bigcap\Theta$ in Θ

$$\Phi(V^\perp) \leq 1 + K\mathbf{d}^2_{g_E}. \tag{5.6}$$

Now consider the *smooth* metric on Ξ

$$\hat{g} = (1 + K\varrho(\mathbf{r})\mathbf{d}^2_{g_E})^{\frac{2}{N}} g_E, \tag{5.7}$$

where ϱ is a smooth increasing function with value zero on $[1.4R, 1.5R]$ and value one on $[1.6R, 2R]$. Set

$$\breve{g} = \rho(\mathbf{r})\hat{g} + (1-\rho(\mathbf{r}))(\|\varpi^* d\psi\|^*_{g_E})^{\frac{2}{N}} g_E, \tag{5.8}$$

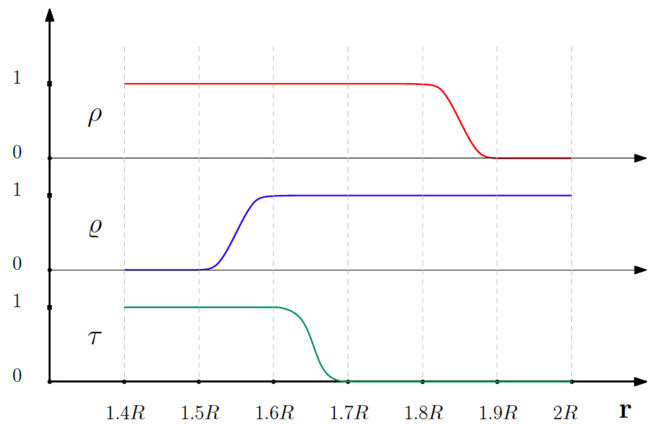
where ρ is one on $[1.4R, 1.8R]$, decreases to zero on $[1.8R, 1.9R]$ and keeps value zero on $[1.9R, 2R]$. On $[1.7R, 2R]$, since $\Phi(V^\perp) = \|\varpi^* d\psi\|^*_{g_E}$, (5.6) guarantees

$$\breve{g} \geq (\|\varpi^* d\psi\|^*_{g_E})^{\frac{2}{N}} g_E.$$

Therefore, on $1.4R \leq \mathbf{r} \leq 2R$,

$$\Phi(V_{\breve{g}}^{\perp}) \leq 1,$$

where $V_{\breve{g}}^{\perp}$ is the the oriented unit N-vector perpendicular to V under $\breve{g}$.



By Lemmas 2.12 and 2.14 in Harvey and Lawson [HL82b] there exists a continuously varying 1-dimensional plane field $\mathscr{W}$ transverse to $V_{\breve{g}}^{\perp}$ for $1.4R \leq \mathbf{r} \leq 2R$ such that under the orthogonal combination $\tilde{g} = \breve{g}|_{V^{\perp}} \oplus \tilde{\alpha}\breve{g}|_{\mathscr{W}}$ for some sufficiently large constant $\tilde{\alpha}$

$$||\Phi||_{\tilde{g}}^{*} = \Phi(V_{\breve{g}}^{\perp}) \leq 1.$$

However a vital flaw is that $\tilde{g}$ may be NOT smooth. To conquer this, note that the angle between V and $\mathscr{W}$ can be assumed strictly less than $\frac{\pi}{4}$ (the angle of V and $\mathscr{W}$ being 0 along $C \bigcap \Xi$) in Ξ on $1.4R \leq \mathbf{r} \leq 2R$. We define a *smooth* metric

$$\bar{g} = \breve{g}|_{V^{\perp}} \oplus [1 + \varrho(\mathbf{r})\rho(\mathbf{r} + 0.1R)\sqrt{2}\tilde{\alpha}]\breve{g}|_{V}$$

on Ξ. (The shift term $0.1R$ is in fact not necessary.) Since

$$\begin{aligned}
&\text{on } [1.4R, 1.6R], && ||\Phi||_{\bar{g}}^{*} \leq ||\Phi||_{\breve{g}}^{*} = ||\phi||_{\breve{g}}^{*} \leq ||\phi||_{g_E}^{*} = 1;\\
&\text{on } [1.6R, 1.7R], && ||\Phi||_{\bar{g}}^{*} \leq ||\Phi||_{\breve{g}}^{*} \leq 1; \text{ and}\\
&\text{on } [1.7R, 2R], && ||\Phi||_{\bar{g}}^{*} \leq ||\Phi||_{\tilde{g}}^{*} = \Phi(V_{\breve{g}}^{\perp}) \leq 1,
\end{aligned}$$

we have

$$||\Phi||_{\bar{g}}^{*} \leq 1.$$

On $[1.4R, 1.5R]$, $\Phi = \phi$ and $\bar{g} = g_E$. Meanwhile, on $[1.9R, 2R]$, $\Phi = \varpi^{*}(d\psi)$ and $\bar{g} = \breve{g} = (||\varpi^{*}d\psi||_{g_E}^{*})^{\frac{2}{N}} g_E$.

It is apparent that this calibration pair of the C^1-calibration Φ and the smooth metric $\bar{g}$ can naturally extend on some neighborhood $\tilde{U}$ of S in our model in §2. According to Theorem 6.2 in [Fed74] S is homologically area-minimizing in $\tilde{U}$.

6. Realization of Oriented Lawlor Cones

Lawlor found many mass-minimizing cones in [Law91] by constructing particular calibrations discontinuous along boundary $\mathfrak{B}$ of some open angular neighborhood $\mathcal{N}$ for each of them. They are of form $\phi = d(f\tilde{\psi})$ where $\tilde{\psi}$ is a smooth $(k-1)$-form on $\overline{\mathcal{N}}$ and where f is at least C^2 along the cone and smooth elsewhere on $\mathcal{N}$, Lipschitzian along $\mathfrak{B}$ with value zero on $(\overline{\mathcal{N}})^c$. Although ϕ is not continuous, through mollifications all oriented cones with such calibrations can be shown mass-minimizing. We will use the same idea.

First, one can similarly follow **Step 1** and **Step 2** in §5 with certain modifications. Here most notations are taken directly from §5.

Recall $\psi = \mathbf{r}\omega$ where ω is the unit volume form of the link L of an oriented Lawlor cone C. Then $d\psi$ is the oriented unit k-dimensional form of $C \sim 0$, and

$$\phi = d(f \cdot \varpi^* \psi)$$

where $f(q) = \tilde{f}(\tan\theta(q))$ and $\theta(q)$ is the angle between $\overrightarrow{Oq}$ and $\overrightarrow{O(\varpi(q))}$. Set $t = \tan\theta(q) = \frac{\mathbf{d}_{g_E}(q)}{\mathbf{r}(q)}$. According to [Law91] $\tilde{f}(t) = 1 - at^2 - bt^3 + \cdots$ near $t = 0$.

Define

$$\Phi = d[\tau(\mathbf{r})(f \cdot \varpi^* \psi) + (1 - \tau(\mathbf{r}))(\varpi^* \psi)].$$

For $q \in \mathcal{N} \sim C$, define $V_q = \frac{\overrightarrow{\varpi(q)q}}{|\overrightarrow{\varpi(q)q}|}$. Then we get a unit vector field V on $\mathcal{N} \sim C$ whose limits on $C \sim 0$ give normal directions of $C \sim 0$. For $q \in \mathcal{N}$, denote by $F_q^\perp$ the oriented unit k-vector perpendicular to the fiber through q and it gives a k-vector field $F^\perp$ in $\mathcal{N}$. Since $L_V(\varpi^*\psi) = 0$,

$$\begin{aligned}
&L_V\Phi \\
=&L_V d[\tau(\mathbf{r})(f\varpi^*\psi) + (1-\tau(\mathbf{r}))(\varpi^*\psi)] \\
=&d[L_V(\tau(\mathbf{r})(f\varpi^*\psi) + (1-\tau(\mathbf{r}))(\varpi^*\psi))] \\
=&d[f\tau(\mathbf{r})L_V(\varpi^*\psi)] + d[L_V(f)\tau(\mathbf{r})\varpi^*\psi] + d[(1-\tau(\mathbf{r}))L_V(\varpi^*\psi))] \\
=&d[L_V(f)\tau(\mathbf{r})\varpi^*\psi]
\end{aligned} \tag{6.1}$$

Let $\gamma(s) = \exp_p(s\nu)$ for $0 \le s < \epsilon$ where ν is a normal direction at a point p of $C \sim 0$ and ϵ is small enough. So $\gamma'(s) = V_{\gamma(s)}$ for $0 < s < \epsilon$ with $\lim_{s\to 0} V_{\gamma(s)} = \nu$. By Lemma 2.3.2 in [Law91],

$$\lim_{s\to 0}(L_V F^\perp)_{\gamma(s)} = \left(\frac{d}{ds}|_{s=0} \det[I - sh_{ij}^\nu]^{-1}\right) F_p^\perp = 0,$$

where h_{ij}^ν is the second fundamental form at p in normal direction ν. Note that by (6.1)

$$\lim_{s\to 0}(L_V\Phi)_{\gamma(s)}$$

involves a normal direction. Therefore

$$\lim_{s\to 0}(L_V[\Phi(F^\perp)])_{\gamma(s)} = 0.$$

Hence there exists a positive constant K such that in a sufficiently small neighborhood Ξ of $C \bigcap \Theta$ in Θ

$$\Phi(F^{\perp}) \leq 1 + K\mathbf{d}^2_{g_E}. \tag{6.2}$$

Then following the procedures in §5 one can obtain a pair of Φ and $\bar{g}$ on some neighborhood $\tilde{U}$ of S, such that

(1). $\bar{g}$ is a smooth metric,
(2). the comass of Φ is no larger than 1 where it is defined, and
(3). Φ is the oriented volume form of the cone along $C \sim 0$.

Take a smaller neighborhood Y of S where $Y \Subset \tilde{U}$ and $(\overline{Y}, \bar{g}|_{\overline{Y}})$ forms a manifold with boundary. Isometrically embed $\overline{Y}$ into some Euclidean space $(\mathbb{R}^s, g_E)$ thru F. By the compactness of $F(\overline{Y})$ there is $\tau > 0$ such that the exponential map restricted to the τ-disk normal bundle $\mathfrak{D}$ over $F(Y)$ is a diffeomorphism. Denote by $\mathfrak{N}$ the image of $\mathfrak{D}$ and by π the induced projection. Choose an open neighborhood $W \Subset Y$ of S. Let $\lambda = \mathrm{dist}_{g_E}(\partial\overline{F(Y)}, \partial\overline{F(W)})$. Then mollify $\pi^*((F^{-1})^*(\Phi))$ with averaging radius $\epsilon < \epsilon_0 = \frac{1}{2}\min\{\lambda, \tau\}$ in the region $\{x \in \mathfrak{N} : \mathrm{dist}_{g_E}(x, \partial\overline{\mathfrak{N}}) \geq \epsilon_0\}$ of $\mathbb{R}^s$. Denote the generated smooth forms by $\tilde{\Phi}_\epsilon$ and set $\Phi_\epsilon = F^*(\tilde{\Phi}_\epsilon|_{F(W)})$. By the commutativity of the exterior differentiation and mollification in $\mathbb{R}^s$, it follows $d\Phi_\epsilon = 0$.

Now we show that S is homologically area-minimizing in $(\overline{W}, \bar{g}|_{\overline{W}})$. By [FF60] there exists a minimizer $T = \overrightarrow{T} \cdot \|T\|$ in $[S]$. Note that, except a measure 0 set $\mathscr{S}$, $\mathbf{spt}T$ is a disjoint union of countably many C^1 submanifolds (see [Fed69]) and denote the bad set $(\mathbf{spt}T \sim \mathscr{S}) \bigcap \mathfrak{B} \sim 0$ by $\mathscr{B}$. Then $\mathscr{B} = \mathscr{C} \coprod \mathscr{O}$ where $\mathscr{C} = \{x \in \mathscr{B} : \overrightarrow{T_x} \in \wedge^k T_x \mathfrak{B}\}$ and $\mathscr{O} = \mathscr{B} \sim \mathscr{C}$. The decomposition is unique up to a $\|T\|$-measure 0 set. Obviously $\mathscr{O}$ is of $\|T\|$-measure 0. Although Φ is not well defined along $\mathfrak{B}$, $\Phi_x(\overrightarrow{T}_x)$ makes sense on $\mathbf{spt}T \sim (\mathscr{S} \bigcup \mathscr{O})$ with value 0 on $\mathscr{C}$ (due to the construction of ϕ in [Law91]). Also note that the uniformly bounded real-valued measurable function sequence $\Phi_\epsilon(\overrightarrow{T})$ converges to $\Phi(\overrightarrow{T})$ pointwise on $\mathbf{spt}T \sim (\mathscr{S} \bigcup \mathscr{O})$ (i.e., almost $\|T\|$-everywhere). Applying Lebesgue's bounded convergence theorem we have

$$\mathbf{M}(S) = \int_S \Phi = \lim_{\epsilon \downarrow 0} \int_S \Phi_\epsilon = \lim_{\epsilon \downarrow 0} \int \Phi_\epsilon(\overrightarrow{T}) d\|T\| = \int \Phi(\overrightarrow{T}) d\|T\| \leq \mathbf{M}(T).$$

Remark 6.1. *Φ_ϵ for $0 < \epsilon < \epsilon_0$ may have comass greater than one under $\bar{g}$.*

Remark 6.2. *Similar argument shows that all Cheng's examples of homogeneous area-minimizing cones of codimension 2 in* [Che88] *(e.g. minimal cones over $U(7)/U(1) \times SU(2)^3$ in $\mathbb{R}^{42}$, $Sp(n) \times Sp(3)/Sp(1)^3 \times Sp(n-3)$ in $\mathbb{R}^{12n}$ for $n \geq 4$, and $Sp(4)/Sp(1)^4$ in $\mathbb{R}^{27}$) can be realized as well.*

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REFERENCES

[All72] William K. Allard, *On the first variation of a varifold*, Ann. Math. **95** (1972), 417–491.

[Che88] Benny N. Cheng, *Area-minimizing cone-type surfaces and coflat calibrations*, Indiana Univ. Math. J. **37** (1988), 505–535.

[FF60] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, Ann. Math. **72** (1960), 458–520.

[Fed69] Herbert Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[Fed74] ———, *Real flat chains, cochains and variational problems*, Indiana Univ. Math. J. **24** (1974), 351–407.

[HS85] Robert Hardt and Leon Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine. Angew. Math. **362** (1985), 102–129.

[HL82a] F. Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[HL82b] ———, *Calibrated foliations*, Amer. J. Math. **104** (1982), 607–633.

[Law91] Gary R. Lawlor, *A Sufficient Criterion for a Cone to be Area-Minimizing,* Vol. 91, Mem. of the Amer. Math. Soc., 1991.

[Law72] H. Blaine Lawson, Jr., *The equivariant Plateau problem and interior regularity*, Trans. Amer. Math. Soc. **173** (1972), 231-249.

[Mor86] Frank Morgan, *On finiteness of the number of stable minimal hypersurfaces with a fixed boundary*, Indiana Univ. Math. J. **35** (1986), 779-833.

[Nas56] John Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.

[Sim83] Leon Simon, *Lectures on Geometric Measure Theory*, Vol. 3, Proc. Centre Math. Anal. Austral. Nat. Univ., 1983.

[Sma99] Nathan Smale, *Singular homologically area minimizing surfaces of codimension one in Riemannian manifolds*, Invent. Math. **135** (1999), 145-183.

[Sma00] ———, *A construction of homologically area minimizing hypersurfaces with higher dimensional singular sets*, Trans. Amer. Math. Soc. **352** (2000), 2319-2330.

[Zhaa] Yongsheng Zhang, *On extending calibration pairs. Available at arXiv:1511.03953.*

[Zhab] ———, *On Lawson's area-minimizing hypercones. Available at arXiv:1501.04681.*

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